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Grand antiprism and quaternions

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Abstract

Vertices of the four-dimensional (4D) semi-regular polytope, the *grand antiprism* and its symmetry group of order 400 are represented in terms of quaternions with unit norm. It follows from the icosian representation of the E_8 root system which decomposes into two copies of the root system of H_4 . The symmetry of the *grand antiprism* is a maximal subgroup of the Coxeter group $W(H_4)$. It is the group $\text{Aut}(H_2 \oplus H'_2)$ which is constructed in terms of 20 quaternionic roots of the Coxeter diagram $H_2 \oplus H'_2$. The root system of H_4 represented by the binary icosahedral group I of order 120, constitutes the regular 4D polytope 600-cell. When its 20 quaternionic vertices corresponding to the roots of the diagram $H_2 \oplus H'_2$ are removed from the vertices of the 600-cell the remaining 100 quaternions constitute the vertices of the *grand antiprism*. We give a detailed analysis of the construction of the cells of the *grand antiprism* in terms of quaternions. The dual polytope of the *grand antiprism* has also been constructed.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The six regular four-dimensional (4D) polytopes and their semi-regular Archimedean polytopes are very interesting because their symmetries $W(A_4)$, $W(B_4)$, $W(F_4)$ and $W(H_4)$ can be represented with the use of finite subgroups of quaternions. There is a close correspondence between the finite subgroups of quaternions and the symmetries of the above polytopes [1]. Some of them occur in high energy physics in model building either as a gauge symmetry like $SU(5)$ [2] or as a little group like $SO(9)$ [3] of the M-theory. The Coxeter group $W(H_4)$ arises as the symmetry group of the polytope 600-cell, $\{3, 3, 5\}$ [4], vertices of which can be represented by 120 quaternions of the binary icosahedral group [5, 6]. The dual polytope 120-cell, $\{5, 3, 3\}$ with 600 vertices, can be constructed from 600-cell in terms of quaternions.

Another observation is that they are all nested in the root system of E_8 represented by icosians [7, 8].

The *grand antiprism* was first constructed in 1965 by Conway and Guy [9] with a computer analysis. An antiprism is defined to be a polyhedron composed of two parallel copies of some particular n -sided polygon, connected by an alternating band of triangles. In this paper we study the symmetry group of the *grand antiprism* and construct its 100 vertices in terms of quaternions. We explicitly show how these vertices form 20 pentagonal antiprisms and 300 tetrahedra. We organize the paper as follows. In section 2 we briefly discuss the correspondence between the root system of H_4 and the quaternionic representation of the binary icosahedral group I and decompose it as $120 = 20 + 100$, where the first 20 quaternions represent the root system of the Coxeter diagram $H_2 \oplus H'_2$ and the remaining quaternions are the vertices of the *grand antiprism*. The symmetry group of the *grand antiprism* turns out to be the group $\text{Aut}(H_2 \oplus H'_2)$ of order 400 which can be constructed directly from the quaternionic roots of the Coxeter diagram $H_2 \oplus H'_2$. A simple technique has been employed in section 3 for the construction of the cell structures of the *grand antiprism*. A projection of the grand antiprism into a 3D space is discussed in section 4. The dual polytope of the *grand antiprism* has been constructed in section 5 where 320 vertices decompose as $320 = 20 + 200 + 100$ under the group $\text{Aut}(H_2 \oplus H'_2)$ where 20 vertices are scaled copies of the root system of the Coxeter diagram $H_2 \oplus H'_2$ lying on a sphere S^3 with radius $\frac{1+\sqrt{5}}{2\sqrt{2}} \approx 1.14$ and 300 vertices lie on an inner sphere S^3 with a radius of unit length. The conclusion is given in section 6 and an appendix lists the decompositions of the vertices of the Archimedean $W(H_4)$ solids in terms of the orbits of $\text{Aut}(H_2 \oplus H'_2)$.

2. Construction of the root system of $H_2 \oplus H'_2$ as the maximal subset of the root system of H_4 and the vertices of the *grand antiprism*

Let $q = q_0 + q_i e_i$ ($i = 1, 2, 3$) be a real quaternion with its conjugate defined by $\bar{q} = q_0 - q_i e_i$ where the quaternionic imaginary units satisfy the relations

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k \quad (i, j, k = 1, 2, 3). \tag{1}$$

Here δ_{ij} and ε_{ijk} are the Kronecker and Lévi-Civita symbols respectively and summation over the repeated indices is implicit. Quaternions generate the 4D Euclidean space where the quaternionic scalar product is defined by

$$(p, q) = \frac{1}{2}(\bar{p}q + \bar{q}p) = \frac{1}{2}(p\bar{q} + q\bar{p}). \tag{2}$$

The imaginary quaternionic units e_i can be related to the Pauli matrices σ_i by $e_i = -i\sigma_i$ and the unit quaternion is represented by the 2×2 unit matrix. This correspondence proves that the group of quaternions is isomorphic to $SU(2)$ which is a double cover of the proper rotation group $SO(3)$. The root system of the Coxeter diagram H_4 can be represented by the quaternionic elements of the binary icosahedral group I [10] which are given as the set of conjugacy classes in table 1. The sets of conjugacy classes are indeed the orbits of the Coxeter group

$$W(H_3) = \{[p, \bar{p}] \oplus [p, \bar{p}]^*, p \in I\} \tag{3}$$

where we use the notation [8]

$$[a, b] : q \rightarrow aqb \quad \text{and} \quad [a, b]^* : q \rightarrow a\bar{q}b. \tag{4}$$

In table 1, $\tau = \frac{1+\sqrt{5}}{2}$, $\sigma = \frac{1-\sqrt{5}}{2}$, and they satisfy the relations $\tau^2 = \tau + 1$, $\sigma^2 = \sigma + 1$, $\tau + \sigma = 1$, $\tau\sigma = -1$.

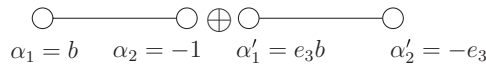


Figure 1. Coxeter diagram of $H_2 \oplus H'_2$ with quaternionic simple roots.

Let $b = \frac{1}{2}(\tau + \sigma e_1 + e_2) \in 12(1)_+$ and $b^5 = -1$ represent the two simple roots of the Coxeter group H_2 with unit norm. Then the quaternions $e_3 b = \frac{1}{2}(-e_1 + \sigma e_2 + \tau e_3)$ and $e_3 b^5 = -e_3$ represent the simple roots of the Coxeter diagram H'_2 orthogonal to the roots of H_2 . The Coxeter diagram of $H_2 \oplus H'_2$ with its simple roots are depicted in figure 1.

The reflection generators at these simple roots are given by [11]

$$[b, -b]^*, \quad [1, -1]^*, \quad [e_3 b, -e_3 b]^*, \quad [e_3, -e_3]^*. \tag{5}$$

These generators will lead to the following 20 quaternionic roots of the Coxeter diagram $H_2 \oplus H'_2$:

$$H_2 \oplus H'_2 = \{b^m, e_3 b^m\} \quad (m = 0, 1, \dots, 9). \tag{6}$$

This set of quaternions forms a maximal subgroup, the dicyclic group of order 20, in the binary icosahedral group I [4]. The generators in (5) generate the group $W(H_2) \times W(H'_2)$, each of which, is a dihedral group of order 10. It is more appropriate to represent each dihedral group with one rotation generator and one reflection generator which can be written as

$$W(H_2) = \{[b, b], [1, -1]^*\}; \quad W(H'_2) = \{[\bar{b}, b], [e_3, -e_3]^*\}. \tag{7}$$

The direct product group is of order 100. One may simply note that the generator $[e_3, 1]$ or $[1, e_3]$ permutes the simple roots of $H_2 \oplus H'_2$. Actually these generators leave the Cartan matrix of the diagram $H_2 \oplus H'_2$ invariant. Without loss of generality we take the generator $[e_3, 1]$ as the new generator which generates the cyclic group C_4 of order 4. The group C_4 is the normalizer of the group $W(H_2) \times W(H'_2)$ in the Coxeter group $W(H_4)$. An extension of the group $W(H_2) \times W(H'_2)$ by the group C_4 is a semi-direct product of the two groups which is the group

$$\text{Aut}(H_2 \oplus H'_2) \approx \{W(H_2) \times W(H'_2)\} : C_4 \tag{8}$$

of order 400 and is one of the five maximal subgroups of the Coxeter group $W(H_4)$ [11].

The group elements now can be written more concisely as

$$\text{Aut}(H_2 \oplus H'_2) = \{[p, q] \oplus [p, q]^*; p, q \in (H_2 \oplus H'_2)\}. \tag{9}$$

It is represented by the pairs of quaternions taking values from the root system of the Coxeter diagram $H_2 \oplus H'_2$. This is the quaternionic representation of the ionic diminished Coxeter group [10, 2⁺, 10] [6].

3. Construction of vertices of the grand antiprism

The set of roots of the Coxeter graph $H_2 \oplus H'_2$ represents the vertices of two decagons in two orthogonal planes sharing only the origin as the common point. For a better clarification let us define a new set of quaternionic units:

$$1' = 1, \quad e'_1 = \frac{\sigma e_1 + e_2}{\sqrt{2 + \sigma}}, \quad e'_2 = e_3 e'_1 = \frac{-e_1 + \sigma e_2}{\sqrt{2 + \sigma}}, \quad e'_3 = e_3. \tag{10}$$

Then the set of roots in (6) can be written as

$$\left\{ \exp\left(\frac{\pi}{5} e'_1\right)^m, e_3 \exp\left(\frac{\pi}{5} e'_1\right)^m \right\} \quad (m = 0, 1, \dots, 9). \tag{11}$$

Table 1. Conjugacy classes of the binary icosahedral group I represented by quaternions for the cases of 1 being fixed. (Sets of conjugacy classes of N elements fixing the elements q are denoted as $N(q)$ where additional subscripts are used to distinguish different sets of the same type.)

Orders of elements	The sets $N(1)$ of the conjugacy classes denoted with N elements
1	1
2	-1
10	$12(1)_+ :$ $\frac{1}{2}(\tau \pm e_1 \pm \sigma e_3), \frac{1}{2}(\tau \pm e_2 \pm \sigma e_1),$ $\frac{1}{2}(\tau \pm e_3 \pm \sigma e_2)$
5	$12(1)_- :$ $\frac{1}{2}(-\tau \pm e_1 \pm \sigma e_3), \frac{1}{2}(-\tau \pm e_2 \pm \sigma e_1),$ $\frac{1}{2}(-\tau \pm e_3 \pm \sigma e_2)$
10	$12(1)'_+ :$ $\frac{1}{2}(\sigma \pm e_1 \pm \tau e_2), \frac{1}{2}(\sigma \pm e_2 \pm \tau e_3),$ $\frac{1}{2}(\sigma \pm e_3 \pm \tau e_1)$
5	$12(1)'_- :$ $\frac{1}{2}(-\sigma \pm e_1 \pm \tau e_2), \frac{1}{2}(-\sigma \pm e_2 \pm \tau e_3),$ $\frac{1}{2}(-\sigma \pm e_3 \pm \tau e_1)$
6	$20(1)_+ :$ $\frac{1}{2}(1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(1 \pm \tau e_1 \pm \sigma e_2),$ $\frac{1}{2}(1 \pm \tau e_2 \pm \sigma e_3), \frac{1}{2}(1 \pm \tau e_3 \pm \sigma e_1)$
3	$20(1)_- :$ $\frac{1}{2}(-1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(-1 \pm \tau e_1 \pm \sigma e_2),$ $\frac{1}{2}(-1 \pm \tau e_2 \pm \sigma e_3), \frac{1}{2}(-1 \pm \tau e_3 \pm \sigma e_1)$
4	$30(1) :$ $\pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm \sigma e_1 \pm \tau e_2 \pm e_3),$ $\frac{1}{2}(\pm \sigma e_2 \pm \tau e_3 \pm e_1), \frac{1}{2}(\pm \sigma e_3 \pm \tau e_1 \pm e_2)$

This indicates that the roots of the first H_2 and the origin define a plane, say the plane **A**, generated by the unit vectors $1, e'_1$ and the second set of roots with the origin define the plane **B**, generated by the unit vectors e'_2 and e_3 . The group element $[b, b]$ acts as a rotation of order 5 in the plane **A** and the element $[\bar{b}, b]$ acts as a rotation of order 5 in the plane **B**. In what follows we prove that the remaining set of 100 quaternions obtained by deleting those quaternions in (6) from the set I , can be written compactly as vertices of the grand antiprism (GA):

$$GA = \{b^m c b^n, b^m e_3 c b^n\} \quad (m, n = 0, 1, \dots, 9). \tag{12}$$

Here $c = \frac{1}{2}(\tau - \sigma e_1 + e_2) \in 12(1)_+$ is one of those nearest quaternions to b in I . In reference [10] we showed that the sets of quaternions in table 1 represent the vertices of some polyhedra possessing the icosahedral symmetry $W(H_3)$. Each set of 12 elements represents an icosahedron, each set of 20 elements represents a dodecahedron and the set with 30 elements represents an icosidodecahedron. The unit quaternions ± 1 represent the poles of the sphere S^3 . It is clear that these polyhedra have the vertices at the intersections of the parallel hyperplanes orthogonal to the unit quaternion 1 with the sphere S^3 . For example, the set $12_+(1)$ can be plotted in a space where the imaginary unit quaternions represent the ordinary x, y, z axes. This means we are in the space represented by the intersection of the sphere S^3 with the hyperplane orthogonal to the unit quaternion 1 and at a distance $\frac{\tau}{2}$ from the origin.

In order to understand the 600-cell, the set $12(1)_+$ plays the most important role. The quaternions in the set $12(1)_+$ constitute the vertices of an icosahedron with the shortest edge length $-\sigma$. They can be decomposed into 20 sets of triads [10] where each triad represents an equilateral triangle. The edge length between the unit quaternion 1 and any one of those quaternions in the set $12(1)_+$ is also $-\sigma$ so that the quaternion 1 and the quaternions of the



Figure 2. The icosahedron of the quaternions of equation (13).

set $12(1)_+$ constitute 20 tetrahedra with a common vertex represented by 1. Table 1 is just one decomposition out of 120 different decompositions where any quaternion $q \in I$ can be chosen instead of 1. This is because the group $W(H_3)$ can be embedded in 120 different ways in the group $W(H_4)$. In each case the group $W(H_3)$ leaves one quaternion, q say, invariant which can be represented by the conjugate group $W(H_3)^q = \{[I, \bar{q}\bar{I}q] \oplus [I, q\bar{I}q]^*\}$ [10]. The 20 tetrahedra having the quaternion q as a common vertex would be represented by the set of quaternions q and $q(12(1)_+) = (12(1)_+)q = 12(q)_+$. This proves that the 600-cell consists of $\frac{20 \times 120}{4} = 600$ tetrahedra.

The set $12(1)_+$ can be written in terms of the quaternions b and c as follows:

$$\begin{array}{cccccc}
 & & b & & & \\
 & c & \bar{b}cb & \bar{b}^2cb^2 & b^2c\bar{b}^2 & bc\bar{b} \\
 c\bar{b} & \bar{b}c & \bar{b}^2cb & -b^2cb^2 & bc\bar{b}^2 & \\
 & & \bar{b} & & &
 \end{array} \tag{13}$$

Indeed the third line is the set of five quaternions $(\bar{b}^mcb^m, m = \{0, 1, 2, 3, 4\})$ obtained by rotating the quaternion \bar{c} around the quaternion b . This assertion can be checked through the relations

$$b^2cb^2 = -\bar{c}, \quad \bar{b}cb = cb\bar{c}, \quad \bar{c}bc = bc\bar{b}, \quad e_3b = \bar{b}e_3, \quad e_3c = \bar{c}e_3. \tag{14}$$

As we stated earlier they are the vertices of an icosahedron when plotted in the ordinary 3D space represented by x, y, z as depicted in figure 2.

If one imagines that the quaternions b and \bar{b} represent the top and the bottom vertices of the icosahedron in (13), then those quaternions in the second line constitute the vertices of a pentagon nearest to the quaternion b and those in the third line represent the bottom pentagonal plane nearest to the quaternion \bar{b} . One can easily show that the quaternions representing the edges of the pentagons in both top and bottom planes can be written as follows:

$$-\sigma(\alpha e'_2 + \beta e_3), \quad \alpha^2 + \beta^2 = 1. \tag{15}$$

Indeed these two planes are parallel to the plane **B** which is orthogonal to the plane **A** generated by 1 and e'_1 . Since b and \bar{b} are removed from the set of quaternions in (13) what remains is the pentagonal antiprism with ten vertices which is depicted in figure 3.

There is now a simple mechanism to generate ten pentagonal antiprisms which form a ring. Before proceeding further we note that all quaternions on the second line of (13) are obtained from the vertex c by a rotation in the plane **B** by an angle 72° . Similarly the quaternions on the third line of (13) are obtained by a 72° rotation of \bar{c} in the plane parallel to the plane **B**.

This shows that the edge length of the pentagon is $-\sigma$.

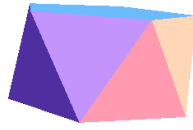


Figure 3. Pentagonal antiprism represented by the quaternionic vertices $(\bar{b}^m cb^m, \bar{b}^m c b^m)$ ($m = 0, 1, 2, 3, 4$).

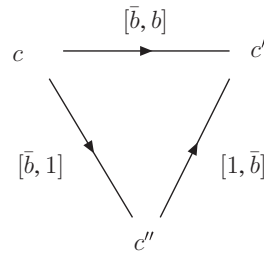


Figure 4. Diagrammatic representation of some of the group elements of $\text{Aut}(H_2 \oplus H_2)$.

Now we discuss how to obtain those 50 quaternions constituting the first ring of ten pentagonal antiprisms. Let us consider the diagram in figure 4.

This shows that the group element $[\bar{b}, b]$ rotates the quaternions by an angle 72° in the plane **B**, in other words, in the ten planes parallel to **B**. The group element $[\bar{b}, 1]$ transforms a given quaternion in the ‘upper plane’ to a quaternion in the ‘lower plane’ which is already rotated by 36° degrees with respect to the ‘upper plane’. The group element $[1, \bar{b}]$ lifts one element from the bottom plane to the upper plane after a 36° rotation. It is interesting to note that just as the sum of these vectors $(c'' - c) + (c' - c'') = c' - c$, the group elements satisfy a similar relation, namely $[\bar{b}, 1][1, \bar{b}] = [\bar{b}, b]$. Inverses of these group elements reverse the directions of the arrows. The quaternions $c, c' = \bar{b}cb, c'' = \bar{b}c$ form an equilateral triangle. Applying $[1, \bar{b}]$ ten times on any quaternion will bring it back to the original quaternion so that they define a decagon. Similarly the action of $[\bar{b}, 1]$ on any quaternion ten times will generate another plane of decagon. It is clear that we obtain the following ring of ten pentagonal antiprisms provided the quaternions b^m ($m = 0, 1, \dots, 9$) are removed from the set I .

Figure 5 indicates that when starting with the quaternion c we obtain the set of 50 quaternions:

$$R_1 = \{b^m cb^n\} \quad (m, n = 0, 1, \dots, 9). \tag{16}$$

It is clear from figure 5 that the vertices of two pentagonal antiprisms in which the quaternion c is a common vertex can be written as follows:

$$\begin{aligned} &cb, \quad \bar{b}cb^2, \quad -\bar{b}^2c\bar{b}^2, \quad b^2c\bar{b}, \quad bc \\ &c, \quad \bar{b}cb, \quad \bar{b}^2cb^2, \quad b^2c\bar{b}^2, \quad bc\bar{b} \\ &\bar{b}c, \quad \bar{b}^2cb, \quad -b^2cb^2, \quad bc\bar{b}^2, \quad c\bar{b}. \end{aligned} \tag{17}$$

Note that the quaternions at the upper two lines can be obtained from those at the lower two lines by a right multiplication by the quaternion b . In other words, the set of quaternions in the upper two lines are the elements of the set $12_+(b)$ from which 1 and b^2 are deleted. They

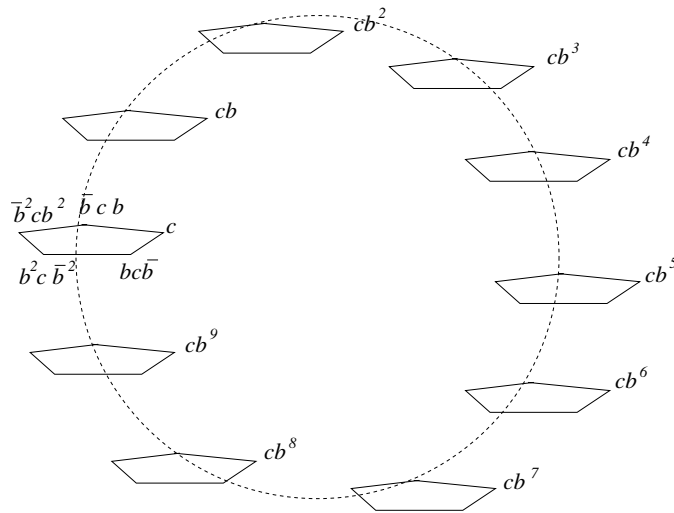


Figure 5. One ring of pentagonal antiprism. The quaternions at the vertices of the pentagons are obtained by $\frac{2\pi}{5}$ rotations around the vertex b .

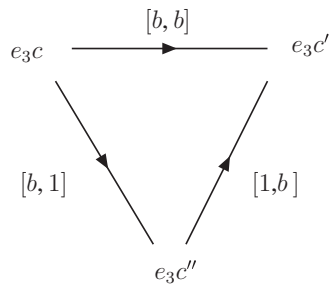


Figure 6. Diagrammatic representation of the action of some group elements on the set of vertices of the second ring of the pentagonal antiprisms.

constitute the pentagonal antiprism in the space of intersection of the hyperplane orthogonal to the quaternion b and the sphere S^3 . We note that the quaternions at the bottom line are the quaternionic conjugates of the quaternions at the middle line. Therefore, the center of the lower pentagonal antiprism is at the point $\frac{\tau}{2}1$ and the center of the upper pentagonal antiprism is at the quaternion $\frac{\tau}{2}b$. If we keep multiplying the quaternions representing a given pentagonal antiprism by b , we create other quaternionic vertices. The next layer of vertices is the set of $12_+(b^2)$ with the omission of b and b^3 where the center is represented by $\frac{\tau}{2}b^2$. We have altogether ten pentagonal antiprisms in this ring represented by the sets $12_+(b^m)$, where the vertices $b^m, m = 0, 1, \dots, 9$, are deleted from the sets and the centers of the pentagonal antiprisms are represented by ten quaternions $\frac{\tau}{2}b^m$. We note that the action of the element $[1, b]$ corresponds to the rotation of the hyperplane whose intersection with the sphere S^3 is a pentagonal antiprism.

When we act by the group element $[e_3, 1]$ on the set of quaternions in the diagram of figure 4 and find the conjugate elements of the group elements we obtain the diagram in figure 6.

A similar consideration will lead to the set of quaternions

$$R_2 = \{b^m e_3 c b^n\} \quad (m, n = 0, 1, \dots, 9), \quad (18)$$

which represents the second ring of vertices of ten pentagonal antiprisms similar to those depicted in figure 5. The two sets of rings satisfy the relations

$$e_3 R_1 = R_1 e_3 = R_2, \quad e_3 R_2 = R_2 e_3 = R_1. \quad (19)$$

It is clear from these discussions that the centers of the pentagonal antiprisms in both rings are represented by the set of 20 quaternions:

$$\frac{\tau}{2} b^m, \quad \frac{\tau}{2} e_3 b^m \quad (m = 0, 1, \dots, 9). \quad (20)$$

The *grand antiprism* has 300 tetrahedral cells, details of which we discuss below. In order to understand the cell structure of a polytope one should check first the cell structures surrounding a given vertex, say, vertex c . We have already seen that the vertex c is surrounded by two pentagonal antiprisms illustrated in equation (17). To understand the nature of the tetrahedra linked to the vertex c we look at the set of quaternions $12_+(c) = 12_+(1)c = c(12_+(1))$ which can be written as follows:

$$\begin{aligned} &cb \\ &c^2, \quad c\bar{b}cb, \quad c\bar{b}^2cb^2, \quad b, \quad cbc\bar{b} \\ &c\bar{b}c, \quad c\bar{b}^2cb, \quad 1, \quad cbc\bar{b}^2, \quad c^2\bar{b} \\ &c\bar{b}. \end{aligned} \quad (21)$$

We note that we should remove two quaternions 1 and b from the set of (21) since they are the roots of H_2 . It is easy to see that the remaining 10 vertices form 12 equilateral triangles. When they are connected to the vertex c we obtain 12 tetrahedra. This proves that the vertex c is surrounded by two pentagonal antiprisms and 12 tetrahedra. Let us write those ten quaternions explicitly:

$$\begin{aligned} q_1 &= \frac{1}{2}(1 + e_1 + e_2 - e_3), & q_2 &= \frac{1}{2}(1 + \tau e_2 - \sigma e_3), \\ q_3 &= \frac{1}{2}(\tau + e_1 - \sigma e_3), & q_4 &= \frac{1}{2}(1 + \tau e_1 - \sigma e_2), \\ q_5 &= \frac{1}{2}(\tau - \sigma e_2 + e_3), & q_6 &= \frac{1}{2}(-\sigma + e_1 + \tau e_2), \\ q_7 &= \frac{1}{2}(\tau + e_1 + \sigma e_3), & q_8 &= \frac{1}{2}(\tau - \sigma e_2 - e_3), \\ q_9 &= \frac{1}{2}(1 + e_1 + e_2 + e_3), & q_{10} &= \frac{1}{2}(1 + \tau e_2 + \sigma e_3). \end{aligned} \quad (22)$$

They belong to the following sets of rings of pentagonal antiprisms:

$$(c, q_2, q_3, q_5, q_7, q_8, q_{10}) \in R_1; \quad (q_1, q_4, q_6, q_9) \in R_2. \quad (23)$$

The 12 tetrahedra can be classified as follows.

Four tetrahedra:

$$(c, q_2; q_6, q_9), \quad (c, q_3; q_4, q_9), \quad (c, q_7; q_1, q_4), \quad (c, q_{10}; q_1, q_9) \quad (24)$$

Six tetrahedra:

$$\begin{aligned} &(c, q_2, q_5; q_9), \quad (c, q_2, q_{10}; q_6), \quad (c, q_3, q_5; q_9), \quad (c, q_3, q_7; q_4), \\ &(c, q_7, q_8; q_1), \quad (c, q_8, q_{10}; q_1) \end{aligned} \quad (25)$$

Two tetrahedra:

$$(c; q_1, q_4, q_6), \quad (c; q_4, q_6, q_9). \quad (26)$$

It is clear from these considerations that the first four tetrahedra in (24) have vertex structures such that two vertices are on the ring R_1 and two vertices are on the ring R_2 . The

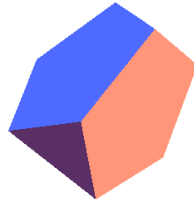


Figure 7. Dissected icosahedron (vertex figure of the *grand antiprism*).

next six tetrahedra in (25) have vertex structures such that three vertices forming an equilateral triangle on the ring R_1 and one vertex on the ring R_2 . Similarly the last two tetrahedra in (26) have the following structure: one vertex on the ring R_1 and three vertices on the ring R_2 . When we consider the tetrahedral structures formed by the quaternion e_3c and the set of quaternions $12_+(e_3c)$, the above structures will be reversed because multiplication by the quaternion e_3 reverses the sets of rings $e_3 : R_1 \leftrightarrow R_2$. When we consider the 24 tetrahedra together we see that they split as $24 = 8 + 8 + 8$. Since for one vertex of the *grand antiprism* we have 12 tetrahedra and each tetrahedron has four vertices then the *grand antiprism* has $\frac{12 \times 100}{4} = 300$ tetrahedra which split as $300 = 100 + 200$. This shows that 100 tetrahedra have the structure such that two vertices are on the ring R_1 and two vertices are on the ring R_2 . The 200 tetrahedra have the structure such that each of 100 tetrahedra has three vertices on the ring R_1 and the fourth vertex on the ring R_2 and each of the next 100 tetrahedra has one vertex on the ring R_1 and three vertices on the ring R_2 . The ten quaternions in (22) constitute the vertices of the vertex figure of the grand antiprism (see figure 7) as they are the nearest vertices to the vertex c . They are on the sphere S^2 determined as the intersection of the sphere S^3 and the hyperplane orthogonal to the quaternion c . To plot the vertices in figure 7 in 3D we introduce a new set of orthogonal basis of quaternionions defined by

$$\begin{aligned} d_0 &= c = \frac{1}{2}(\tau - \sigma e_1 + e_2), & d_1 &= e_1c = \frac{1}{2}(\sigma + \tau e_1 + e_3), \\ d_2 &= e_2c = \frac{1}{2}(-1 + \tau e_2 + \sigma e_3), & d_3 &= e_3c = \frac{1}{2}(-e_1 - \sigma e_2 + \tau e_3). \end{aligned} \tag{27}$$

In the new basis a quaternion q would read

$$q = q_0d_0 + q_1d_1 + q_2d_2 + q_3d_3 \equiv (q_0, q_1, q_2, q_3)$$

and the ten quaternions above all have first components equal to $\frac{\tau}{2}$. If their first components are deleted and an overall factor $\frac{1}{2}$ is omitted then they can be written as

$$(0, \pm\sigma, \pm 1), \quad (-\sigma, \pm 1, 0), \quad (\sigma, 1, 0), \quad (1, 0, \pm\sigma), \quad (-1, 0, \sigma). \tag{28}$$

These are the vertices of the solid called *dissected icosahedron* shown in figure 7.

The quaternions 1 and b which were deleted can be represented by $(-1, 0, \sigma)$, $(\sigma, -1, 0)$ respectively in the new basis. With the inclusion of these two vertices to the vertices in (28) one would obtain an icosahedron [12].

Before we end this section we determine the symmetry of the vertex figure. The subgroup of the group $\text{Aut}(H_2 \oplus H_2')$ fixing the quaternion c , in other words, leaving the hyperplane orthogonal to c invariant is the symmetry of the vertex figure. Therefore, it suffices to determine the subgroup which leaves the quaternion c invariant. One can check that it is the group generated by two commuting group elements $[e_3, -e_3]^*$ and $[\bar{b}^2, -\bar{b}^2]^*$. So it is Klein's four-group $C_2 \times C_2$ of order 4. The vertices of the *dissected icosahedron* which are left invariant under the action of the group $C_2 \times C_2$ are given as sets of quaternions in the brackets as follows:

$$(q_1, q_9), \quad (q_2, q_3, q_7, q_{10}), \quad (q_4, q_6), \quad (q_5, q_8). \tag{29}$$

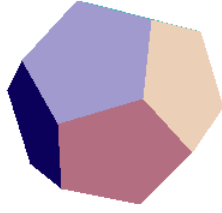


Figure 8. Dodecahedron obtained from the projection of the *grand antiprism*.

4. Projection of the grand antiprism into a 3D space

Let us consider the intersection of the hyperplanes orthogonal to the unit quaternion 1 with the 3D sphere S^3 of unit radius consisting of 120 vertices of the 600-cell displayed in table 1. When the set of vectors $b^m, e_3 b^m$ ($m = 0, 1, 2, \dots, 9$) are removed from the set of quaternions of table 1, we obtain several pentagonal antiprisms some of which are not regular. The regular pentagonal prisms follow from the sets $12_{\pm}(1)$ and $12'_{\pm}(1)$ when $\pm b, \pm \bar{b}, \pm b^2, \pm \bar{b}^2$ are removed from the sets. The vertices of four pentagonal antiprisms can be compactly written as

$$\begin{aligned} (\bar{b}^m c b^m, \bar{b}^m \bar{c} b^m), & \quad (-\bar{b}^m c b^m, -\bar{b}^m \bar{c} b^m), \\ (\bar{b}^m c^2 b^m, \bar{b}^m \bar{c}^2 b^m), & \quad (-\bar{b}^m c^2 b^m, -\bar{b}^m \bar{c}^2 b^m), \end{aligned} \quad (m = 0, 1, 2, 3, 4). \tag{30}$$

They are in the intersections of the sphere S^3 with the parallel hyperplanes orthogonal to the unit quaternion 1 and with real coordinates at the distances $\pm \frac{\tau}{2}$ and $\pm \frac{\sigma}{2}$. One of these pentagonal antiprism has been depicted in figure 3. Similarly, at distances $\pm \frac{1}{2}$ from the origin along the quaternion 1 we have two dodecahedra, namely the sets 20_+ and 20_- one of which is depicted in figure 8.

One of the sets, say, 20_+ now decomposes into two sets as $20_+ = 10_+ + 10'_+$ under a subgroup, of order 20, of the group $\text{Aut}(H_2 \oplus H'_2)$. It is the group $C_2 \times W(H'_2)$ which leaves the unit quaternion 1 invariant. The generators of the group can be written as $C_2 \times W(H'_2) = \{[1, 1]^*, [\bar{b}, b], [e_3, -e_3]^*\}$. The other set 20_- has a similar decomposition. Each set of quaternions 10_+ and $10'_+$ can be written as

$$\begin{aligned} 10_+ &= \left\{ \frac{1}{2}(1 + e_1 + e_2 + e_3), \frac{1}{2}(1 - e_1 - e_2 - e_3), \right. \\ &\quad \frac{1}{2}(1 - e_1 - e_2 + e_3), \frac{1}{2}(1 + e_1 + e_2 - e_3), \\ &\quad \left. \frac{1}{2}(1 \pm \sigma e_1 \pm \tau e_3), \frac{1}{2}(1 + \tau e_1 - \sigma e_2), \frac{1}{2}(1 - \tau e_1 + \sigma e_2) \right\}, \\ 10'_+ &= \left\{ \frac{1}{2}(1 - e_1 + e_2 - e_3), \frac{1}{2}(1 + e_1 - e_2 - e_3), \right. \\ &\quad \frac{1}{2}(1 - e_1 + e_2 + e_3), \frac{1}{2}(1 + e_1 - e_2 - e_3), \\ &\quad \left. \frac{1}{2}(1 \pm \tau e_2 \pm \sigma e_3), \frac{1}{2}(1 + \tau e_1 + \sigma e_2), \frac{1}{2}(1 - \tau e_1 - \sigma e_2) \right\}. \end{aligned} \tag{31}$$

These are the distorted pentagonal antiprisms depicted in figure 9.

The intersection of the hyperplane orthogonal to the quaternion 1 with the sphere S^3 at the origin will lead to 20 vertices of the *grand antiprism* given by

$$\begin{aligned} \pm e_1, \pm e_2, \frac{1}{2}(\pm \sigma e_1 \pm \tau e_2 \pm e_3), \pm \frac{1}{2}(e_1 + \sigma e_2 + \tau e_3), \\ \pm \frac{1}{2}(e_1 + \sigma e_2 - \tau e_3), \pm \frac{1}{2}(-\tau e_1 + e_2 + \sigma e_3), \pm \frac{1}{2}(\tau e_1 - e_2 + \sigma e_3). \end{aligned} \tag{32}$$

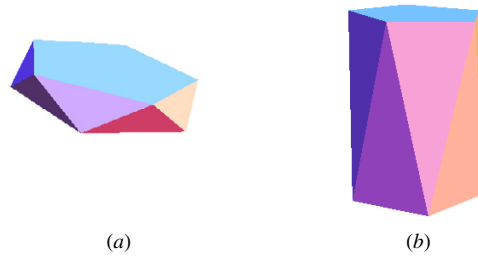


Figure 9. Distorted pentagonal antiprisms obtained from a dodecahedron (a) the solid represented by the set of vertices of 10_+ and (b) the solid represented by the vertices of $10'_+$.

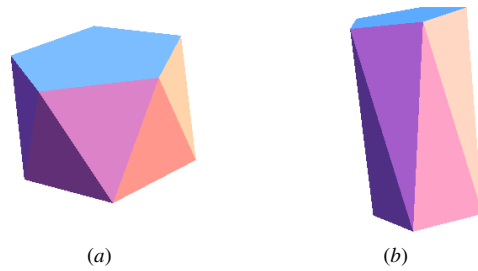


Figure 10. (a) Pentagonal antiprism represented by the vertices in (33) (first line), (b) The solid representing the vertices of (33) (second line).

These are the remaining quaternions of the set 30 of table 1 after removing ten quaternions $\{e_3 b^m\}$ ($m = 0, 1, \dots, 9$).

Under the group $C_2 \times W(H'_2)$ the set of quaternions in (32) splits as

$$\begin{aligned}
 P_1 &= \left\{ \pm e_1, \pm \frac{1}{2}(e_1 + \sigma e_2 + \tau e_3), \pm \frac{1}{2}(e_1 + \sigma e_2 - \tau e_3), \right. \\
 &\quad \left. \pm \frac{1}{2}(-\sigma e_1 + \tau e_2 \pm e_3) \right\}, \\
 P_2 &= \left\{ \pm e_2, \pm \frac{1}{2}(\tau e_1 - e_2 + \sigma e_3), \pm \frac{1}{2}(-\tau e_1 + e_2 + \sigma e_3), \right. \\
 &\quad \left. \pm \frac{1}{2}(\sigma e_1 + \tau e_2 \pm e_3) \right\}.
 \end{aligned} \tag{33}$$

The two sets in (33) are pentagonal antiprism and non-regular stretched antiprism, respectively as plotted in figure 10.

5. Dual of the *grand antiprism*

To determine one cell of the *dual grand antiprism* one should find the centers of the cells which share one vertex of the *grand antiprism*. Let us choose a particular vertex represented by the quaternion c . We have seen that there are 2 pentagonal antiprisms and 12 tetrahedra sharing the quaternion c as a common vertex. We have already noted that the centers of two pentagonal antiprisms are $\frac{\tau}{2}1$ and $\frac{\tau}{2}b$. The centers of the 12 tetrahedra can be determined by

averaging the corresponding vertices. The centers of the tetrahedra, up to a scale factor, are given by the following unit quaternions:

$$\begin{aligned}
 c_1 &= \frac{1}{2\sqrt{2}}(\tau + e_1 + 2e_2 - \sigma e_3), & c_2 &= \frac{1}{2\sqrt{2}}(\tau - \sigma e_1 + \sqrt{5}e_2), \\
 c_3 &= \frac{1}{2\sqrt{2}}(2 - \sigma e_1 + \tau e_2 + e_3), & c_4 &= \frac{1}{2\sqrt{2}}(2 + \tau e_1 + e_2 + \sigma e_3), \\
 c_5 &= \frac{1}{2\sqrt{2}}(\sqrt{5} + \tau e_1 - \sigma e_2), & c_6 &= \frac{1}{2\sqrt{2}}(\sqrt{5} + e_1 + e_2 - e_3), \\
 c_7 &= \frac{1}{2\sqrt{2}}(\tau + \tau e_1 + \tau e_2 - \sigma^2 e_3), & c_8 &= \frac{1}{2\sqrt{2}}(\tau + e_1 + 2e_2 + \sigma e_3), \\
 c_9 &= \frac{1}{2\sqrt{2}}(\tau + \tau e_1 + \tau e_2 + \sigma^2 e_3), & c_{10} &= \frac{1}{2\sqrt{2}}(2 + \tau e_1 + e_2 - \sigma e_3), \\
 c_{11} &= \frac{1}{2\sqrt{2}}(\sqrt{5} + e_1 + e_2 + e_3), & c_{12} &= \frac{1}{2\sqrt{2}}(2 - \sigma e_1 + \tau e_2 - e_3).
 \end{aligned} \tag{34}$$

One can easily check that any vector $c_i - c_j$ ($i \neq j = 1, 2, \dots, 12$) is orthogonal to the vector c . One notes that the quaternion $1 - b$ is also orthogonal to the quaternion c . However, the quaternions $1 - c_i$ and $b - c_i$ are not orthogonal to the quaternion c . The relative magnitudes of 1 and b must be fixed to have their tips on the same hyperplane with the vectors c_i . One can easily check that the quaternions $c_{13} = \frac{\tau}{\sqrt{2}}1$ and $c_{14} = \frac{\tau}{\sqrt{2}}b = \frac{1}{2\sqrt{2}}(\tau + \sigma e_1 + e_2)$ satisfy the requirement. These 14 quaternions are in a single hyperplane which is orthogonal to the vertex c . The 14 quaternions c_i ($i = 1, 2, \dots, 14$) represent the vertices of the cell of the *dual grand antiprism*. It is clear that the vertices of the *dual grand antiprism* have two different lengths, 20 of which are of length $\frac{\tau}{\sqrt{2}}$ and the remaining 300 vertices are of unit length. Therefore, vertices of the *dual grand antiprism* lie on two concentric spheres S^3 with radii $\frac{\tau}{\sqrt{2}}$ and 1. The cell can be plotted, similar to the vertex figure, in the space of intersections of the hyperplane, orthogonal to the quaternion c , with two S^3 spheres. This means we need to use the basis introduced in (27). When 14 vertices c_i ($i = 1, 2, \dots, 14$), are expressed in terms of the new basis vectors then all first components would read $\frac{\tau^2}{2\sqrt{2}}d_0$ and the remaining three components, up to a scale factor $\frac{1}{2\sqrt{2}}$, can be written as

$$\begin{aligned}
 c_1 &\approx (-\sigma, -\sigma, -\sigma), & c_2 &\approx (0, 1, \sigma^2), \\
 c_3 &\approx (\sigma^2, 0, 1), & c_4 &\approx (\sigma^2, 0, -1), \\
 c_5 &\approx (-\sigma, \sigma, \sigma), & c_6 &\approx (-\sigma^2, 0, -1) \\
 c_7 &\approx (-\sigma, -\sigma, \sigma), & c_8 &\approx (0, 1, -\sigma^2), \\
 c_9 &\approx (1, \sigma^2, 0) & c_{10} &\approx (1, -\sigma^2, 0), \\
 c_{11} &\approx (-\sigma, \sigma, -\sigma), & c_{12} &\approx (\sigma, -\sigma, \sigma), \\
 c_{13} &\approx (-1, -\tau, 0), & c_{14} &\approx (-\tau, 0, 1).
 \end{aligned} \tag{35}$$

We obtain the following solid in three dimensions as shown in figure 11.

This cell has the same symmetry $C_2 \times C_2$ of the vertex figure which fixes the quaternion c . The following sets of vertices of the cell are left invariant under the action of $C_2 \times C_2$:

$$(c_1, c_4, c_8, c_{10}), \quad (c_2, c_5), \quad (c_3, c_6, c_{11}, c_{12}), \quad (c_7, c_9), \quad (c_{13}, c_{14}). \tag{36}$$

One cell of the *dual grand antiprism* consists of four regular pentagons, four kites and two isosceles trapezoids where the edge lengths are shown in figure 12.

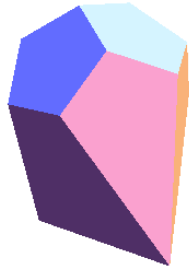


Figure 11. One cell of the *dual grand antiprism*.

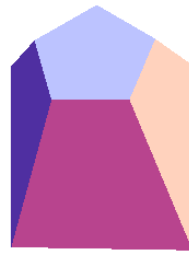


Figure 12. Faces of the cell of *dual grand antiprism*.

To visualize the cell of the *dual grand antiprism* we note that the following sets of vertices form the faces of the cell:

$$\begin{aligned}
 &\text{pentagons :} \\
 &(c_1, c_2, c_8, c_7, c_9), \quad (c_8, c_{12}, c_6, c_4, c_7), \\
 &(c_7, c_4, c_5, c_{10}, c_9), \quad (c_9, c_{10}, c_{11}, c_3, c_1); \\
 &\text{kites :} \\
 &(c_{14}, c_2, c_1, c_3), \quad (c_{14}, c_2, c_8, c_{12}), \quad (c_{13}, c_5, c_4, c_6), \quad (c_{13}, c_{11}, c_{10}, c_5); \\
 &\text{isoscelestrapezoids :} \\
 &(c_{13}, c_{14}, c_{12}, c_6), \quad (c_{13}, c_{14}, c_3, c_{11}).
 \end{aligned} \tag{37}$$

Let us note that the pair of vertices $(c_{13}, c_{14}) \in H_2$ is left invariant by a group larger than the group $C_2 \times C_2$. Indeed the largest group fixing the set $(c_{13}, c_{14}) \in H_2$ is the direct product of two groups $C_2 \times W(H'_2)$ where the first group is generated by the group element $[\bar{b}^2, -\bar{b}^2]^* \in W(H_2)$. The generators of the group $C_2 \times W(H'_2)$ can be chosen as follows:

$$C_2 \times W(H'_2) = \{[\bar{b}, b], [e_3, -e_3]^*, [\bar{b}^2, -\bar{b}^2]^*\}. \tag{38}$$

The center of the cell of the *dual grand antiprism* represented by the vertex c , as we know, is left invariant by two generators in (38). The generator $[\bar{b}, b]$ generates the five vertices $\bar{b}^m c b^m$ ($m = 0, 1, 2, 3, 4$) of the *grand antiprism* when applied to c . Action of the group element $[\bar{b}, b]$ on the other vertices c_i ($i = 1, 2, \dots, 12$) will generate other cells joined to the vertices $(c_{13}, c_{14}) \in H_2$. That is to say we generate five cells all sharing the vertices c_{13} and c_{14} . For each pair of quaternions (b^m, b^{m+2}) ($m = 0, 1, \dots, 9$) we have $5 \times 10 = 50$ cells overall and $(e_3 b^m, e_3 b^{m+2})$ ($m = 0, 1, \dots, 9$) another 50 cells with a total of 100 cells of the *dual grand antiprism*. This is, of course, the number of vertices of the *grand antiprism*. As we noted before, the dual grand antiprism has 20 vertices belonging to the root system $H_2 \oplus H'_2$.

The remaining 300 vertices belong to two different orbits of $\text{Aut}(H_2 \oplus H'_2)$ of sizes 200 and 100. They can be computed as follows. In the earlier paper [10] we determined the 600 vertices of the 120-cell as the set of quaternions $J = \sum_{i,j=0}^5 b^i T^j b^j$ where $T' = V_1 \oplus V_2 \oplus V_3$ represents the vertices of the 24-cell with V_1, V_2, V_3 given by

$$\begin{aligned} V_1 &= \left\{ \frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3) \right\}, \\ V_2 &= \left\{ \frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1) \right\}, \\ V_3 &= \left\{ \frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2) \right\}. \end{aligned} \tag{39}$$

It is not difficult to see that the 600 vertices of the 120-cell decompose under the group $\text{Aut}(H_2 \oplus H'_2)$ as $600 = 200 + 200 + 100 + 100$. With the choice of $b = \frac{1}{2}(\tau + \sigma e_1 + e_2)$, we see that the 300 vertices of the *dual grand antiprism* are given by the set of quaternions

$$\begin{aligned} J_1 &= \sum_{i,j=0}^5 b^i V_1 b^j \text{ of size 200,} \\ J'_3 &= \sum_{i,j=0}^5 b^i \frac{1}{\sqrt{2}}(\pm 1 \pm e_3) b^j \text{ of size 100.} \end{aligned} \tag{40}$$

One can check that in (35) the first 12 vertices belong to the set of 20 vertices of a dodecahedron and form 4 pentagonal faces only instead of 12 pentagonal faces of a dodecahedron. This is because eight vertices are missing from the dodecahedron which is a typical cell of the 120-cell. For each vertex of the *grand antiprism* we have 12 vertices forming 4 pentagons which is part of the dual cell, then the number of vertices of the *dual grand antiprism* belonging to the set J is $\frac{100 \times 12}{4} = 300$ as we have explained before which is the union of the sets J_1 and J'_3 .

6. Conclusion

We have constructed the *grand antiprism* and its dual polytope with quaternions as a subset of the vertices of the 600-cell represented by the quaternionic elements of the binary icosahedral group. It is a 4D semi-regular polytope with 100 vertices and 320 cells made of 300 tetrahedral and 20 pentagonal antiprisms. Construction of its vertices and its symmetry group elements in terms of quaternions is very simple and very elegant. One needs only two orthogonal quaternions b and e_3 to obtain the group elements and an additional quaternion c for the construction of the vertices. The originality of the work lies in the fact that both the vertices of the grand antiprism and its symmetry group are constructed in terms of quaternions. We also note that the dual polytope of the grand antiprism has not been constructed elsewhere.

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Appendix. Decomposition of the semi-regular orbits of $W(H_4)$ under its maximal subgroup $\text{Aut}(H_2 \oplus H'_2) \approx \{W(H_2) \times W(H'_2)\} : C_4$

Orbit 1:
 $600 = 2(100) + 2(200).$

Orbit 2:

$$1200 = 3(200) + 2(100) + 400.$$

Orbit 3:

$$720 = 100 + 3(200) + 20.$$

Orbit 4:

$$120 = 20 + 100.$$

Orbit 5:

$$3600 = 6(400) + 5(200) + 2(100).$$

Orbit 6:

$$2400 = 6(200) + 3(400).$$

Orbit 7:

$$3600 = 5(200) + 6(400) + 2(100).$$

Orbit 8:

$$1440 = 5(200) + 400 + 40.$$

Orbit 9:

$$2400 = 3(400) + 6(200).$$

Orbit 10:

$$3600 = 6(400) + 5(200) + 2(100).$$

Orbit 11:

$$7200 = 6(200) + 15(400).$$

Orbit 12:

$$7200 = 15(400) + 6(200).$$

Orbit 13:

$$7200 = 15(400) + 6(200).$$

Orbit 14:

$$7200 = 15(400) + 6(200).$$

Orbit 15:

$$14400 = 36(400).$$

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